QUANTUM MATRIX BALL: THE WEIGHTED BERGMAN KERNELS

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The Cartan domains are among the important subjects in many problems of representation theory and mathematical physics [1, 4]. The methods of quantum groups theory [2] were used in [7] to produce q-analogues of Cartan domains, in particular, q-analogues of balls in the spaces of complex matrices.

The point of this work is to consider those quantum matrix balls and the associated Hilbert spaces of 'functions'. As a main result, we present an explicit formula for the weighted Bergman kernel.

It is implicit everywhere in the sequel that $q \in (0,1)$, $m,n \in \mathbb{N}$, and $m \le n$.

We need q-analogues for $*$-algebras $\mathrm{Pol}(\mathrm{Mat}_{mn})$ of polynomials on the space Mat_{mn} of complex matrices and a $*$-algebra $D(\mathbb{U})$ of smooth finite functions in the matrix ball $\mathbb{U} = \{A \in \mathrm{Mat}_{mn} |\ \|A\| < 1\}$. We start with forming a q-analogue for the algebra $\mathrm{Fun}(\mathbb{U}) = \mathrm{Pol}(\mathrm{Mat}_{mn}) + D(\mathbb{U})$.

The $*$-algebra $\mathrm{Fun}(\mathbb{U})_q$ is given by its generators f_0, z_a^α, $a = 1,2,\ldots,n$, $\alpha = 1,2,\ldots,m$, and the relations

$$\begin{cases} z_a^\alpha z_b^\beta = q z_b^\beta z_a^\alpha & , \quad a=b \quad \& \quad \alpha<\beta \qquad \text{or} \qquad a<b \quad \& \quad \alpha=\beta \\ z_a^\alpha z_b^\beta = z_b^\beta z_a^\alpha & , \quad \alpha<\beta \quad \& \quad a>b \\ z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha = (q-q^{-1}) z_a^\beta z_b^\alpha & , \quad \alpha<\beta \quad \& \quad a<b \end{cases} \tag{1}$$

$$\left(z_b^\beta\right)^* \cdot z_a^\alpha = q^2 \sum_{a',b'=1}^{n} \sum_{\alpha',\beta'=1}^{m} R(b,a,b',a') R(\beta,\alpha,\beta',\alpha') \cdot z_{a'}^{\alpha'} \cdot \left(z_{b'}^{\beta'}\right)^* + (1-q^2)\delta_{ab}\delta^{\alpha\beta}, \tag{2}$$

$$(z_a^\alpha)^* f_0 = f_0 z_a^\alpha = 0, \tag{3}$$

$$f_0 = f_0^* = f_0^2.$$

Here $a,b = 1,2,\ldots,n$, $\alpha,\beta = 1,2,\ldots,m$,

$$R(i,j,i',j') = \begin{cases} q^{-1} & , \quad i \ne j \quad \& \quad i=i' \quad \& \quad j=j' \\ 1 & , \quad i=j=i'=j' \\ -(q^{-2}-1) & , \quad i=j \quad \& \quad i'=j' \quad \& \quad j'>j \\ 0 & , \quad \textit{otherwise} \end{cases}.$$

In this setting, the $*$-subalgebra $\mathrm{Pol}(\mathrm{Mat}_{mn})_q \subset \mathrm{Fun}(\mathbb{U})_q$ generated by z_a^α, $a = 1,2,\ldots,n$, $\alpha = 1,2,\ldots,m$ is a q-analogue of the $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_{mn})$, and the bilateral ideal $D(\mathbb{U})_q =$

$\mathrm{Pol}(\mathrm{Mat}_{mn})_q f_0 \mathrm{Pol}(\mathrm{Mat}_{mn})_q$ is a q-analogue of the $*$-algebra $D(\mathbb{U})$. (The element f_0 works here as a delta-function, as one can see from (3)).

To motivate our subsequent constructions, observe that (see [6]) the $*$-algebra $D(\mathbb{U})_q$ is a $U_q\mathfrak{su}_{nm}$-module algebra [2]. Remind the explicit formula for invariant integral from [6].

Consider the representation T of $\mathrm{Fun}(\mathbb{U})_q$ in the space $\mathcal{H} = \mathrm{Fun}(\mathbb{U})_q f_0 = \mathrm{Pol}(\mathrm{Mat}_{mn})_q f_0$:

$$T(f)\psi = f\psi, \qquad f \in \mathrm{Fun}(\mathbb{U})_q, \quad \psi \in \mathcal{H}.$$

There exists a unique positive scalar product in $\mathcal{H}$ such that $(f_0, f_0) = 1$, and

$$(T(f)\psi_1, \psi_2) = (\psi_1, T(f^*)\psi_2), \qquad f \in \mathrm{Fun}(\mathbb{U})_q, \quad \psi_1, \psi_2 \in \mathcal{H}.$$

One can prove that the $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_{mn})_q$ admits a unique up to unitary equivalence *faithful* irreducible $*$-representation by *bounded* operators in a Hilbert space. This $*$-representation can be produced via extending the operators $T(f)$, $f \in \mathrm{Pol}(\mathrm{Mat}_{mn})_q$, onto the completion of the pre-Hilbert space $\mathcal{H}$.

The invariant integral is of the form (see [6]):

$$\int\limits_{\mathbb{U}_q} f d\nu = \mathrm{tr}(T(f) q^{-2\Gamma(\check{\rho})})), \qquad f \in D(\mathbb{U})_q, \tag{4}$$

with $\Gamma : \mathfrak{h} \to \mathrm{End}(\mathcal{H})$ being a subrepresentation of the natural representation of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_N$ in $\mathrm{Fun}(\mathbb{U})_q$, and $\check{\rho} \in \mathfrak{h}$ the element of this Cartan subalgebra determined by the half sum of positive roots ρ under the standard pairing of $\mathfrak{h}$ and $\mathfrak{h}^*$. (To see that this integral is well defined, observe that the operators $T(f)$, $f \in D(\mathbb{U})_q$, are finite dimensional, and $\mathcal{H}$ is decomposable into a sum of weight subspaces associated to non-negative weights.)

Our immediate intention is to produce q-analogues of weighted Bergman spaces. In the case $q = 1$ one has

$$\det(1 - \mathbf{z}\mathbf{z}^*) = 1 + \sum_{k=1}^{m} (-1)^k \mathbf{z}^{\wedge k} \mathbf{z}^{*\wedge k}, \tag{5}$$

with $\mathbf{z}^{\wedge k}$, $\mathbf{z}^{*\wedge k}$ being the "exterior powers" of the matrices $\mathbf{z}$, $\mathbf{z}^*$, that is, matrices formed by the minors of order k. The operators $(1-q^2)^{-1/2} T(z_a^\alpha)$, $(1-q^2)^{-1/2} T((z_a^\alpha)^*)$, $a = 1, 2, \ldots, n$, $\alpha = 1, 2, \ldots, m$, are respectively the q-analogues of creation and annihilation operators. The "creation operators" are placed in the right hand side of (5) to the left of the "annihilation operators". This allows one to produce a q-analogue of the polynomial $\det(1 - \mathbf{z}\mathbf{z}^*)$ in a standard way as follows.

Let $1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_k \le m$, $1 \le a_1 < a_2 < \ldots < a_k \le n$. Introduce q-analogues of minors for the matrix $\mathbf{z}$:

$$\mathbf{z}^{\wedge k}{}^{\{\alpha_1, \alpha_2, \ldots, \alpha_k\}}_{\{a_1, a_2, \ldots, a_k\}} = \sum_{s \in S_k} (-q)^{l(s)} z_{a_1}^{\alpha_{s(1)}} z_{a_2}^{\alpha_{s(2)}} \ldots z_{a_k}^{\alpha_{s(k)}},$$

with $l(s) = \mathrm{card}\{(i,j)|\ i < j \quad \& \quad s(i) > s(j)\}$ being the length of the permutation s.

The q-analogue $y \in \mathrm{Pol}(\mathrm{Mat}_{mn})_q$ for the polynomial $\det(1 - \mathbf{z}\mathbf{z}^*)$ is defined by

$$y = 1 + \sum_{k=1}^{m} (-1)^k \sum_{\{J'|\ \mathrm{card}(J')=k\}} \sum_{\{J''|\ \mathrm{card}(J'')=k\}} \mathbf{z}^{\wedge k}{}^{J'}_{J''} \cdot \left(\mathbf{z}^{\wedge k}{}^{J'}_{J''}\right)^*.$$

Let $\lambda > m+n-1$. Now (4) allows one to define the integral with weight y^λ as follows:

$$\int\limits_{\mathbb{U}_q} f d\nu_\lambda \overset{\text{def}}{=} C(\lambda)\mathrm{tr}(T(f)T(y)^\lambda q^{-2\Gamma(\check{\rho})}), \qquad f \in D(\mathbb{U})_q,$$

with $C(\lambda) = \prod\limits_{j=0}^{n-1}\prod\limits_{k=0}^{m-1}(1 - q^{2(\lambda+1-N)}q^{2(j+k)})$ being a normalizing multiple that provides $\int\limits_{\mathbb{U}_q} 1 d\nu_\lambda = 1$.

The Hilbert space $L^2(d\nu_\lambda)_q$ is defined as a completion of the space $D(\mathbb{U})_q$ of finite functions with respect to the norm $\|f\|_\lambda = \left(\int\limits_{\mathbb{U}_q} f^* f d\nu_\lambda\right)^{1/2}$. The closure $L^2_a(d\nu_\lambda)_q$ in $L^2(d\nu_\lambda)_q$ of the unital subalgebra $\mathbb{C}[\mathrm{Mat}_{mn}]_q \subset \mathrm{Pol}(\mathrm{Mat}_{mn})_q$ generated by z_a^α, $a = 1,2,\dots,n$, $\alpha = 1,2,\dots,m$, will be called a weighted Bergman space.

Note that the relations (1) and the algebra $\mathbb{C}[\mathrm{Mat}_{mn}]_q$ were considered in many works on quantum groups (see [2]). The $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_{mn})_q$ determined by the relations (1) and (2), is a q-analogue of the Weyl algebra. This becomes plausible after one performs the 'change of variables' as follows:

$$z_a^\alpha \mapsto (1-q^2)^{-1/2} z_a^\alpha; \qquad (z_a^\alpha)^* \mapsto (1-q^2)^{-1/2}(z_a^\alpha)^*.$$

Consider the orthogonal projection P_λ in $L^2(d\nu_\lambda)_q$ onto the weighted Bergman space $L^2_a(d\nu_\lambda)_q$. It is possible to show that P_λ could be written as an integral operator (see [8])

$$P_\lambda f = \int\limits_{\mathbb{U}_q} K_\lambda(\mathbf{z}, \boldsymbol{\zeta}^*) f(\boldsymbol{\zeta}) d\nu_\lambda(\boldsymbol{\zeta}), \qquad f \in D(\mathbb{U})_q. \tag{6}$$

Our intention is to introduce the algebra $\mathbb{C}[[\mathrm{Mat}_{mn} \times \overline{\mathrm{Mat}}_{mn}]]_q$ of kernels of integral operators and to determine an explicit form of the weighted Bergman kernel $K_\lambda \in \mathbb{C}[[\mathrm{Mat}_{mn} \times \overline{\mathrm{Mat}}_{mn}]]_q$ involved in (6).

Introduce the notation

$$\mathbb{k}_i = \sum_{\substack{J' \subset \{1,2,\dots,m\} \\ \mathrm{card}(J')=i}} \sum_{\substack{J'' \subset \{1,2,\dots,n\} \\ \mathrm{card}(J'')=i}} z^{\wedge i}{}^{J'}_{J''} \otimes \left(z^{\wedge i}{}^{J'}_{J''}\right)^*. \tag{7}$$

Let $\mathbb{C}[\overline{\mathrm{Mat}}_{mn}]_q \subset \mathrm{Pol}(\mathrm{Mat}_{mn})_q$ be the unital subalgebra generated by $(z_a^\alpha)^*$, $a = 1,2,\dots,n$, $\alpha = 1,2,\dots,m$, and $\mathbb{C}[\mathrm{Mat}_{mn}]_q^{\mathrm{op}}$ the algebra which differs from $\mathbb{C}[\mathrm{Mat}_{mn}]_q$ by a replacement of its multiplication law to the opposite one (this replacement is motivated in [8]). The tensor product algebra $\mathbb{C}[\mathrm{Mat}_{mn}]_q^{\mathrm{op}} \otimes \mathbb{C}[\overline{\mathrm{Mat}}_{mn}]_q$ will be called an algebra of polynomial kernels. It is possible to show that in this algebra $\mathbb{k}_i\mathbb{k}_j = \mathbb{k}_j\mathbb{k}_i$ for all $i,j = 1,2,\dots,m$.

We follow [7, 6] in equipping $\mathrm{Pol}(\mathrm{Mat}_{mn})_q$ with a $\mathbb{Z}$-grading: $\deg(z_a^\alpha) = 1$, $\deg((z_a^\alpha)^*) = -1$, $a = 1,2,\dots,n$, $\alpha = 1,2,\dots,m$. In this context one has:

$$\mathbb{C}[\mathrm{Mat}_{mn}]_q^{\mathrm{op}} = \bigoplus_{i=0}^{\infty} \mathbb{C}[\mathrm{Mat}_{mn}]_{q,i}^{\mathrm{op}}, \qquad \mathbb{C}[\overline{\mathrm{Mat}}_{mn}]_q = \bigoplus_{j=0}^{\infty} \mathbb{C}[\overline{\mathrm{Mat}}_{mn}]_{q,-j}$$

$$\mathbb{C}[\mathrm{Mat}_{mn}]_q^{\mathrm{op}} \otimes \mathbb{C}[\overline{\mathrm{Mat}}_{mn}]_q = \bigoplus_{i,j=0}^{\infty} \mathbb{C}[\mathrm{Mat}_{mn}]_{q,i}^{\mathrm{op}} \otimes \mathbb{C}[\overline{\mathrm{Mat}}_{mn}]_{q,-j} \tag{8}$$

The kernel algebra $\mathbb{C}[[\mathrm{Mat}_{mn} \times \overline{\mathrm{Mat}}_{mn}]]_q$ will stand for a completion of $\mathbb{C}[\mathrm{Mat}_{mn}]_q^{\mathrm{op}} \otimes \mathbb{C}[\overline{\mathrm{Mat}}_{mn}]_q$ in the topology associated to the grading in (8). The kernel algebra is constituted by formal series $\psi = \sum\limits_{i,j=0}^{\infty} \psi_{ij}$, with $\psi_{ij} \in \mathbb{C}[\mathrm{Mat}_{mn}]_{q,i}^{\mathrm{op}} \otimes \mathbb{C}[\overline{\mathrm{Mat}}_{mn}]_{q,-j}$.

Our main result is the following formula for the weighted Bergman kernel:

$$K_\lambda = \prod_{j=0}^{\infty} \left(1 + \sum_{i=1}^{m} (-q^{2(\lambda+j)})^i \mathbb{k}_i\right) \cdot \prod_{j=0}^{\infty} \left(1 + \sum_{i=1}^{m} (-q^{2j})^i \mathbb{k}_i\right)^{-1} \tag{9}$$

with $\mathbb{k}_i$ being the polynomial kernels (7). (The right hand side of (9) determines an element of $\mathbb{C}[[\mathrm{Mat}_{mn} \times \overline{\mathrm{Mat}}_{mn}]]_q$ since $\mathbb{k}_i \in \mathbb{C}[\mathrm{Mat}_{mn}]_{q,i}^{\mathrm{op}} \otimes \mathbb{C}[\overline{\mathrm{Mat}}_{mn}]_{q,-i}$ for all $i = 1, 2, \ldots, m$).

In the special case $m = n = 1$ we get a well known result [5]:

$$K_\lambda = \prod_{j=0}^{\infty} \left(1 - q^{2(\lambda+j)} z \otimes \zeta^*\right) \cdot \left(\prod_{j=0}^{\infty} (1 - q^{2j} z \otimes \zeta^*)\right)^{-1} =$$

$$= \sum_{i=0}^{\infty} \frac{(1 - q^{2\lambda})(1 - q^{2(\lambda+1)}) \ldots (1 - q^{2(\lambda+i-1)})}{(1 - q^2)(1 - q^4) \ldots (1 - q^{2i})} z^i \otimes \zeta^{*i}.$$

Now passage to a limit as $q \to 1$ and replacement of $\otimes$ by a dot yields $K_\lambda \underset{q\to 1}{\to} (1 - z\zeta^*)^{-\lambda}$.

A q-analogue of an ordinary Bergman kernel for the matrix ball (see [3]) is derivable from (9) by a substitution $\lambda = m + n$:

$$K = \prod_{j=0}^{m+n-1} \left(1 + \sum_{i=1}^{m} (-q^{2j})^i \mathbb{k}_i\right)^{-1} \underset{q\to 1}{\to} (\det(1 - \mathbf{z} \cdot \boldsymbol{\zeta}^*))^{-(m+n)}.$$

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